

Convergence Rate of EM Scheme for SDDEs

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Abstract

In this paper we investigate the convergence rate of Euler-Maruyama scheme for a class of stochastic differential *delay* equations, where the corresponding coefficients may be *highly nonlinear* with respect to the delay variables. In particular, we reveal that the convergence rate of Euler-Maruyama scheme is $\frac{1}{2}$ for the Brownian motion case, while show that it is best to use the mean-square convergence for the pure jump case, and that the order of mean-square convergence is close to $\frac{1}{2}$.

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1 Introduction

Since most stochastic differential equations (SDEs) can not be solved explicitly, numerical methods have become essential. Recently, there is extensive literature in investigating the strong convergence, weak convergence or sample path convergence of numerical schemes for SDEs, e.g., in [3] for SDEs with a monotone condition, in [4, 7, 12] for SDEs with jumps, in [6, 7, 9, 10] for stochastic differential delay equations (SDDEs) and in [4, 5] for SDEs with a one-side Lipschitz condition. For the comprehensive monographs on numerical approximate methods of SDEs, we can also refer to [8, 12, 13]. Although the results on convergence of Euler-Maruyama (EM) schemes are substantial, there are limited ones on convergence rate under weaker conditions than global Lipschitz condition and linear growth condition. For example, a recent work in [2] reveals the convergence rate of EM schemes for a class of SDEs under a Hölder condition, and, with local Lipschitz constants satisfying a logarithm growth condition, [15] and [1, 7] discuss the convergence rate of EM approximate methods for SDEs and stochastic functional differential equations with jumps, respectively. We should also

point out that the strong convergence of EM schemes for SDDEs is, in general, discussed under a linear growth condition or bounded moments of analytic and numerical solutions, e.g., [7, 9, 10], and that the convergence rate [1, 7] is also revealed under a linear growth condition.

To further motivate our work, we first consider an SDDE on \mathbb{R}

$$(1.1) \quad dX(t) = \{aX(t) + bX^3(t - \tau)\}dt + cX^2(t - \tau)dW(t),$$

where $a, b, c \in \mathbb{R}, \tau > 0$, are constant, and $W(t)$ is a scalar Brownian motion. It is easy to observe that both the drift coefficient and the diffusion coefficient are *highly nonlinear* especially with respect to the delay arguments. Therefore, the existing convergence results, e.g., [7, 9, 10], can not cover Eq. (1.1), and the convergence rate of the corresponding EM scheme can not also be revealed by the techniques of [1, 7] as we have explained in the end of the last paragraph. On the other hand, our work is also enlightened by the recent work in [2] such that consider SDE on \mathbb{R}

$$dX(t) = \{f(t, X(t)) + g(t, X(t))\}dt + \sigma(t, X(t))dW(t),$$

and discuss the convergence rate of the associated EM method, where g is Hölder continuous, of *linear growth*, and *monotone decreasing with respect to the second variable*.

Motivated by the previous literature, in this paper we not only study the strong convergence of EM schemes for a class of SDDEs, which may be *highly nonlinear* with respect to the delay variables, but also reveal the *convergence rate* of the corresponding EM numerical methods. The rest of the paper are organized as follows: under highly nonlinear growth conditions with respect to the delay arguments, in Section 2 we reveal the convergence rate of EM schemes for SDDEs driven by Brownian motion is $\frac{1}{2}$, while in Section 3 we show that it is best to use the mean-square convergence for the pure jump case, and that the rate of mean-square convergence is close to $\frac{1}{2}$.

2 Convergence Rate for Brownian Motion Case

For integer $n > 0$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$ be the Euclidean space and $\|A\| := \sqrt{\text{trace}(A^*A)}$ the Hilbert-Schmidt norm for a matrix A , where A^* is its transpose. Let $W(t)$ be an m -dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. Throughout the paper, $C > 0$ denotes a generic constant whose values may change from lines to lines.

For fixed $T > 0$, in this section we consider SDDE on \mathbb{R}^n

$$(2.1) \quad dX(t) = b(X(t), X(t - \tau))dt + \sigma(X(t), X(t - \tau))dW(t), \quad t \in [0, T]$$

with initial data $X(\theta) = \xi(\theta), \theta \in [-\tau, 0]$.

To guarantee the existence and uniqueness of solution we introduce the following conditions. Let $V_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$(2.2) \quad V_i(x, y) \leq K_i(1 + |x|^{q_i} + |y|^{q_i}), \quad i = 1, 2$$

for some $K_i > 0, q_i \geq 1$ and arbitrary $x, y \in \mathbb{R}^n$. We further assume that

(A1) $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and there exists $L_1 > 0$ such that

$$|b(x_1, y_1) - b(x_2, y_2)| \leq L_1|x_1 - x_2| + V_1(y_1, y_2)|y_1 - y_2|$$

for $x_i, y_i \in \mathbb{R}^n, i = 1, 2$;

(A2) $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and there exists $L_2 > 0$ such that

$$\|\sigma(x_1, y_1) - \sigma(x_2, y_2)\| \leq L_2|x_1 - x_2| + V_2(y_1, y_2)|y_1 - y_2|$$

for $x_i, y_i \in \mathbb{R}^n, i = 1, 2$,

We now introduce an EM method for Eq. (2.1). Without loss of generality, we may assume that there exist sufficiently large integers $N, M > 0$ such that

$$(2.3) \quad \Delta := \frac{\tau}{N} = \frac{T}{M} \in (0, 1).$$

Define a continuous EM scheme associated with Eq. (2.1)

$$(2.4) \quad dY(t) = b(\bar{Y}(t), \bar{Y}(t - \tau))dt + \sigma(\bar{Y}(t), \bar{Y}(t - \tau))dW(t),$$

where $\bar{Y}(t) := Y(k\Delta)$ for $t \in [k\Delta, (k+1)\Delta), k = 0, 1, \dots, M-1$, and $\bar{Y}(\theta) = \xi(\theta), \theta \in [-\tau, 0]$.

Remark 2.1. Clearly, if b and σ are globally Lipschitzian, then b and σ are satisfied with (A1) and (A2). On the other hand, we remark that b and σ may be *highly nonlinear* with respect to the delay variables. There are many such examples which are covered by (A1) – (A2). For example, for Eq. (1.1) it is trivial to see that $b(x, y) = ax + by^3$, $\sigma(x, y) = cy^2$, and (A1) – (A2) hold by choosing $V_1(x, y) = \frac{3|b|}{2}(x^2 + y^2)$ and $V_2(x, y) = |c|(|x| + |y|)$. In fact, the examples, where the drift coefficient and the diffusion coefficient are polynomial of degree $d \geq 1$ with regard to the delay variables, are included in our framework.

Lemma 2.1. Assume that (A1) and (A2) hold. Then, for any initial data $\xi \in C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$, Eq. (2.1) admits a unique global strong solution $X(t), t \in [0, T]$. Moreover, for any $p \geq 2$ there exists $C > 0$ such that

$$(2.5) \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C,$$

and

$$(2.6) \quad \mathbb{E}|Y(t) - \bar{Y}(t)|^p \leq C\Delta^{\frac{p}{2}}.$$

Proof. Note that Eq. (2.1) has a unique local solution due to the fact that both b and σ are locally Lipschitzian. To verify that Eq. (2.1) admits a unique global solution on time interval $[0, T]$, it is sufficient to show that

$$(2.7) \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \leq C, \quad p \geq 2.$$

By a straightforward computation, we can deduce from (A1), (A2) and (2.2) that

$$(2.8) \quad |b(x, y)| \leq C(1 + |x| + |y| + |y|^{q_1+1}), \quad x, y \in \mathbb{R}^n,$$

and

$$(2.9) \quad \|\sigma(x, y)\| \leq C(1 + |x| + |y| + |y|^{q_2+1}), \quad x, y \in \mathbb{R}^n.$$

Set $\gamma_1 := q_1 + 1$ and $\gamma_2 := q_2 + 1$. To show (2.7), by (2.8) and (2.9), the Hölder inequality and the Burkhold-Davis-Gundy inequality, we have that for any $p \geq 2$ and $t \in [0, T]$

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |X(s)|^p \right) &\leq 3^{p-1} \left\{ |\xi(0)|^p + \mathbb{E} \left(\sup_{0 \leq s \leq t} \left| \int_0^s b(X(r), X(r-\tau)) dr \right|^p \right) \right. \\ &\quad \left. + \mathbb{E} \left(\sup_{0 \leq s \leq t} \left| \int_0^s \sigma(X(r), X(r-\tau)) dW(r) \right|^p \right) \right\} \\ &\leq C \left\{ 1 + \mathbb{E} \int_0^t (|b(X(s), X(s-\tau))|^p + \|\sigma(X(s), X(s-\tau))\|^p) ds \right\} \\ &\leq C \left\{ 1 + \mathbb{E} \int_0^t |X(s)|^p ds + \mathbb{E} \int_0^t (|X(s-\tau)|^{p\gamma_1} + |X(s-\tau)|^{p\gamma_2}) ds \right\}, \end{aligned}$$

where we have also used the Young inequality in the last step. This, together with the Gronwall inequality, yields that for $t \in [0, T]$ and $p \geq 2$

$$(2.10) \quad \mathbb{E} \left(\sup_{0 \leq s \leq t} |X(s)|^p \right) \leq C \left\{ 1 + \mathbb{E} \int_0^t (|X(s-\tau)|^{p\gamma_1} + |X(s-\tau)|^{p\gamma_2}) ds \right\}.$$

The following argument is similar to that of [14, Theorem 2.1], however we give a detailed proof, which will also be used in the proof of Theorem 2.2 below. Let $\beta := \gamma_1 \vee \gamma_2$, and

$$p_i := ([T/\tau] + 2 - i)p\beta^{[T/\tau]+1-i}, \quad i = 1, 2, \dots, [T/\tau] + 1,$$

where $[a]$ denotes the integer part of real number a . Thus, due to $\beta \geq 1$ and $p \geq 2$, it is easy to see that $p_i \geq 2$ such that

$$p_{i+1}\beta < p_i \text{ and } p_{[T/\tau]+1} = p, \quad i = 1, 2, \dots, [T/\tau].$$

By (2.10), together with $\xi \in C_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$, we obtain that

$$\mathbb{E} \left(\sup_{0 \leq s \leq \tau} |X(s)|^{p_1} \right) \leq C,$$

which, combining (2.10) with the Hölder inequality, further leads to

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq 2\tau} |X(s)|^{p_2} \right) &\leq C \left\{ 1 + \mathbb{E} \int_0^{2\tau} (|X(s-\tau)|^{p_2\gamma_1} + |X(s-\tau)|^{p_2\gamma_2}) ds \right\} \\ &\leq C \left\{ 1 + \int_0^\tau \left((\mathbb{E}|X(s)|^{p_1})^{\frac{p_2\gamma_1}{p_1}} + (\mathbb{E}|X(s)|^{p_1})^{\frac{p_2\gamma_2}{p_1}} \right) ds \right\} \\ &\leq C. \end{aligned}$$

Repeating the previous procedures gives (2.7) and $\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C$. Finally, the statement (2.6) can also be obtained by taking into account the Hölder inequality, the Burkhold-Davis-Gundy inequality and (2.5). \square

Remark 2.2. Lemma 2.1 gives a new result on existence and uniqueness of solutions to SDDEs on finite-time interval, where the coefficients may be polynomial of any degree $d \geq 1$ with regard to the delay variables.

We can now state our main result, which not only shows the strong convergence of EM scheme associated with Eq. (2.1) but also reveals its convergence rate, although the drift coefficient and the diffusion coefficient may be highly nonlinear with respect to the delay arguments.

Theorem 2.2. Under (A1) and (A2), for any $p \geq 2$ there exists $C > 0$ such that

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |X(s) - Y(s)|^p\right) \leq C \Delta^{\frac{p}{2}},$$

that is, the rate of convergence of EM scheme (2.4) is $\frac{1}{2}$.

Proof. The argument is motivated by that of [2, Theorem 2.1]. For fixed $\delta > 1$ and arbitrary $\epsilon \in (0, 1)$, there exists a continuous nonnegative function $\psi_{\delta\epsilon}(x), x \geq 0$, with support $[\epsilon/\delta, \epsilon]$, such that

$$\int_{\epsilon/\delta}^{\epsilon} \psi_{\delta\epsilon}(x) dx = 1 \text{ and } \psi_{\delta\epsilon}(x) \leq \frac{2}{x \ln \delta}, \quad x > 0.$$

Define

$$\phi_{\delta\epsilon}(x) := \int_0^x \int_0^y \psi_{\delta\epsilon}(z) dz dy, \quad x > 0.$$

Then $\phi_{\delta\epsilon} \in C^2(\mathbb{R}_+; \mathbb{R}_+)$ possesses the following properties:

$$(2.11) \quad x - \epsilon \leq \phi_{\delta\epsilon}(x) \leq x, \quad x > 0,$$

and

$$(2.12) \quad 0 \leq \phi'_{\delta\epsilon}(x) \leq 1, \quad \phi''_{\delta\epsilon}(x) \leq \frac{2}{x \ln \delta} \mathbf{1}_{[\epsilon/\delta, \epsilon]}(x), \quad x > 0.$$

Define

$$(2.13) \quad V_{\delta\epsilon}(x) := \phi_{\delta\epsilon}(|x|), \quad x \in \mathbb{R}^n.$$

By the definition of $\phi_{\delta\epsilon}$, it is trivial to note that $V_{\delta\epsilon} \in C^2(\mathbb{R}^n; \mathbb{R}_+)$. For any $x \in \mathbb{R}^n$ set

$$(V_{\delta\epsilon})_x(x) := \left(\frac{\partial V_{\delta\epsilon}(x)}{\partial x_1}, \dots, \frac{\partial V_{\delta\epsilon}(x)}{\partial x_n} \right) \text{ and } (V_{\delta\epsilon})_{xx}(x) := \left(\frac{\partial^2 V_{\delta\epsilon}(x)}{\partial x_i \partial x_j} \right)_{n \times n}.$$

We then have

$$\frac{\partial V_{\delta\epsilon}(x)}{\partial x_i} = \phi'_{\delta\epsilon}(|x|) \frac{x_i}{|x|} \text{ and } \frac{\partial^2 V_{\delta\epsilon}(x)}{\partial x_i \partial x_j} = \phi'_{\delta\epsilon}(|x|)(\delta_{ij}|x|^2 - x_i x_j)|x|^{-3} + \phi''_{\delta\epsilon}(|x|) x_i x_j |x|^{-2},$$

for $x \in \mathbb{R}^n$, $i = 1, 2, \dots, n$, where $\delta_{ij} = 1$ if $i = j$ or otherwise 0, and

$$(2.14) \quad 0 \leq |(V_{\delta\epsilon})_x(x)| \leq 1 \text{ and } \|(V_{\delta\epsilon})_{xx}(x)\| \leq 2n \left(1 + \frac{1}{\ln \delta}\right) \frac{1}{|x|} \mathbf{1}_{[\epsilon/\delta, \epsilon]}(|x|), \quad x \in \mathbb{R}^n.$$

For any $t \in [0, T]$, let

$$Z(t) := X(t) - Y(t), \quad \bar{Z}(t) := Y(t) - \bar{Y}(t) \quad \text{and} \quad \tilde{Z}(t) := (X(t), \bar{Y}(t)) \in \mathbb{R}^{2n}.$$

Application of the Itô formula yields that

$$\begin{aligned} V_{\delta\epsilon}(Z(t)) &= \int_0^t \langle (V_{\delta\epsilon})_x(Z(s)), b(X(s), X(s-\tau)) - b(\bar{Y}(s), \bar{Y}(s-\tau)) \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \text{trace}\{(\sigma(X(s), X(s-\tau)) - \sigma(\bar{Y}(s), \bar{Y}(s-\tau)))^* (V_{\delta\epsilon})_{xx}(Z(s)) \\ &\quad \times (\sigma(X(s), X(s-\tau)) - \sigma(\bar{Y}(s), \bar{Y}(s-\tau)))\} ds \\ &\quad + \int_0^t \langle (V_{\delta\epsilon})_x(Z(s)), (\sigma(X(s), X(s-\tau)) - \sigma(\bar{Y}(s), \bar{Y}(s-\tau))) dW(s) \rangle \\ &:= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

By (2.14), (A1) and the Hölder inequality, we derive that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |I_1(s)|^p \right) &\leq \int_0^t \mathbb{E} |b(X(s), X(s-\tau)) - b(\bar{Y}(s), \bar{Y}(s-\tau))|^p ds \\ (2.15) \quad &\leq C \int_0^t \left\{ \mathbb{E} |Z(s)|^p + \left(\mathbb{E} V_1^{2p}(\tilde{Z}(s-\tau)) \right)^{\frac{1}{2}} \left(\mathbb{E} |Z(s-\tau)|^{2p} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \mathbb{E} |\bar{Z}(s)|^p + \left(\mathbb{E} V_1^{2p}(\tilde{Z}(s)) \right)^{\frac{1}{2}} \left(\mathbb{E} |\bar{Z}(s)|^{2p} \right)^{\frac{1}{2}} \right\} ds, \quad t \in [0, T], \end{aligned}$$

and due to (A2) and (2.14) again that

$$\begin{aligned} (2.16) \quad \mathbb{E} \left(\sup_{0 \leq s \leq t} |I_2(s)|^p \right) &\leq C \int_0^t \mathbb{E} \{ \|(V_{\delta\epsilon})_{xx}(Z(s))\| \|\sigma(X(s), X(s-\tau)) - \sigma(\bar{Y}(s), \bar{Y}(s-\tau))\|^2 \}^p ds \\ &\leq C \mathbb{E} \int_0^t \frac{1}{|Z(s)|^p} \{ |Z(s)|^{2p} + V_2^{2p}(\tilde{Z}(s-\tau)) |Z(s-\tau)|^{2p} \\ &\quad + |\bar{Z}(s)|^{2p} + V_2^{2p}(\tilde{Z}(s)) |\bar{Z}(s)|^{2p} \} \mathbf{1}_{[\epsilon/\delta, \epsilon]}(|Z(s)|) ds \\ &\leq C \int_0^t \left\{ \mathbb{E} |Z(s)|^p + \frac{1}{\epsilon^p} \left(\mathbb{E} V_2^{4p}(\tilde{Z}(s-\tau)) \right)^{\frac{1}{2}} \left(\mathbb{E} |Z(s-\tau)|^{4p} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{1}{\epsilon^p} \mathbb{E} |\bar{Z}(s)|^{2p} + \frac{1}{\epsilon^p} \left(\mathbb{E} V_2^{4p}(\tilde{Z}(s)) \right)^{\frac{1}{2}} \left(\mathbb{E} |\bar{Z}(s)|^{4p} \right)^{\frac{1}{2}} \right\} ds, \quad t \in [0, T]. \end{aligned}$$

By virtue of the Burkhold-Davis-Gundy inequality, the Hölder inequality and (2.14), for any $p \geq 2$ and $t \in [0, T]$

$$\begin{aligned}
(2.17) \quad \mathbb{E} \left(\sup_{0 \leq s \leq t} |I_3(s)|^p \right) &\leq C \mathbb{E} \left(\int_0^t \|\sigma(X(s), X(s-\tau)) - \sigma(\bar{Y}(s), \bar{Y}(s-\tau))\|^2 ds \right)^{\frac{p}{2}} \\
&\leq C \mathbb{E} \int_0^t \|\sigma(X(s), X(s-\tau)) - \sigma(\bar{Y}(s), \bar{Y}(s-\tau))\|^p ds \\
&\leq C \int_0^t \left\{ \mathbb{E} |Z(s)|^p ds + \left(\mathbb{E} V_2^{2p}(\tilde{Z}(s-\tau)) \right)^{\frac{1}{2}} \left(\mathbb{E} |Z(s-\tau)|^{2p} \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \mathbb{E} |\bar{Z}(s)|^p + \left(\mathbb{E} V_2^{2p}(\tilde{Z}(s)) \right)^{\frac{1}{2}} \left(\mathbb{E} |\bar{Z}(s)|^{2p} \right)^{\frac{1}{2}} \right\} ds.
\end{aligned}$$

Furthermore, observe from (2.2) and (2.5) that

$$\mathbb{E} V_1^{2p}(\tilde{Z}(s-\tau)) + \mathbb{E} V_2^{4p}(\tilde{Z}(s-\tau)) \leq C$$

and by (2.6) that $\mathbb{E} |\bar{Z}(t)|^p \leq C \Delta^{\frac{p}{2}}$. Then, combining (2.15), (2.16) with (2.17), we thus obtain from (2.11) that, for any $t \in [0, T]$ and any $p \geq 2$,

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq s \leq t} |Z(s)|^p \right) &\leq 2^{p-1} \left\{ \epsilon^p + \mathbb{E} \left(\sup_{0 \leq s \leq t} V_{\delta\epsilon}^p(Z(s)) \right) \right\} \\
&\leq C \left\{ \epsilon^p + \Delta^{\frac{p}{2}} + \frac{1}{\epsilon^p} \Delta^p + \frac{1}{\epsilon^p} \Delta^p \right. \\
&\quad \left. + \int_0^t \mathbb{E} |Z(s)|^p ds + \int_0^t \left(\mathbb{E} |Z(s-\tau)|^{2p} \right)^{\frac{1}{2}} ds + \frac{1}{\epsilon^p} \int_0^t \left(\mathbb{E} |Z(s-\tau)|^{4p} \right)^{\frac{1}{2}} ds \right\}.
\end{aligned}$$

This, together with the Gronwall inequality, implies

$$\begin{aligned}
(2.18) \quad \mathbb{E} \left(\sup_{0 \leq s \leq t} |Z(s)|^p \right) &\leq C \left\{ \epsilon^p + \Delta^{\frac{p}{2}} + \frac{1}{\epsilon^p} \Delta^p + \int_0^t \left(\mathbb{E} |Z(s-\tau)|^{2p} \right)^{\frac{1}{2}} ds \right. \\
&\quad \left. + \frac{1}{\epsilon^p} \int_0^t \left(\mathbb{E} |Z(s-\tau)|^{4p} \right)^{\frac{1}{2}} ds \right\}.
\end{aligned}$$

For any $p \geq 2$, let

$$p_i := ([T/\tau] + 2 - i) p 4^{[T/\tau] + 1 - i}, \quad i = 1, 2, \dots, [T/\tau] + 1.$$

It is easy to see that

$$(2.19) \quad 4p_{i+1} < p_i \quad \text{and} \quad p_{[T/\tau] + 1} = p, \quad i = 1, 2, \dots, [T/\tau].$$

Noting that $Z(s-\tau) = 0$ for $s \in [0, \tau]$ and taking $\epsilon = \Delta^{\frac{1}{2}}$ in (2.18), we obtain that

$$\mathbb{E} \left(\sup_{0 \leq s \leq \tau} |Z(s)|^{p_1} \right) \leq C \Delta^{\frac{p_1}{2}}.$$

This, together with (2.19) and the Hölder inequality, further gives that

$$\begin{aligned}\mathbb{E}\left(\sup_{0 \leq s \leq 2\tau} |Z(s)|^{p_2}\right) &\leq C\left\{\Delta^{\frac{p_2}{2}} + \int_0^{2\tau} \left(\mathbb{E}|Z(s-\tau)|^{p_1}\right)^{\frac{p_2}{p_1}} ds\right. \\ &\quad \left. + \Delta^{-\frac{p_2}{2}} \int_0^{2\tau} \left(\mathbb{E}|Z(s-\tau)|^{p_1}\right)^{\frac{2p_2}{p_1}} ds\right\} \\ &\leq C\Delta^{\frac{p_2}{2}}\end{aligned}$$

by taking $\epsilon = \Delta^{\frac{1}{2}}$ in (2.18). The desired assertion then follows by repeating the previous procedures. \square

Remark 2.3. The strong convergence of EM scheme for SDDEs is generally investigated under local Lipschitz condition and bounded moments of analytic solutions and numerical solutions, or local Lipschitz condition and linear growth condition, e.g., [10]. In this section, for a class of SDDEs, which may be *highly nonlinear* with respect to the delay variables, we show the strong convergence of EM scheme under rather general conditions. To the best of our knowledge, there are relatively few results in the existing literature.

Remark 2.4. There are only limited results on convergence order of EM scheme for SDEs or SDDEs under weaker condition than global Lipschitz and linear growth condition. For example, under a Hölder continuous condition, [2] reveals the convergence order of EM scheme for a class of SDEs, and, with local Lipschitz constants satisfying a logarithm growth condition, [15] and [1, 7] discuss the convergence rate of EM approximate methods for SDEs and stochastic functional differential equations with jumps respectively, where *linear growth condition* is imposed in [1, 7]. While, in this section, under very general conditions we reveal the convergence order of EM scheme for a class of SDDEs although which are *highly nonlinear* with respect to delay arguments.

3 Convergence Rate for Pure Jump Case

In the last section we discuss the strong convergence of EM scheme for a class of SDDEs, and reveal the convergence rate is $\frac{1}{2}$ although both the drift coefficient and the diffusion coefficient may be highly nonlinear with respect to the delay variables. In this section we turn to the counterpart for SDDEs with jumps. We further need to introduce some notation. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ -algebra on \mathbb{R} , and $\lambda(dx)$ a σ -finite measure defined on $\mathcal{B}(\mathbb{R})$. Let $p = (p(t)), t \in D_p$, be a stationary \mathcal{F}_t -Poisson point process on \mathbb{R} with characteristic measure $\lambda(\cdot)$. Denote by $N(dt, du)$ the Poisson counting measure associated with p , i.e., $N(t, U) = \sum_{s \in D_p, s \leq t} I_U(p(s))$ for $U \in \mathcal{B}(\mathbb{R})$. Let $\tilde{N}(dt, du) := N(dt, du) - dt\lambda(du)$ be the compensated Poisson measure associated with $N(dt, du)$. In what follows, we further assume that $\int_U |u|^p \lambda(du) < \infty$ for any $p \geq 2$.

In this section we consider SDDE with jumps on \mathbb{R}^n

$$(3.1) \quad dX(t) = b(X(t), X(t-\tau))dt + \int_U h(X(t), X(t-\tau), u)\tilde{N}(dt, du), \quad t \in [0, T]$$

with initial data $X(\theta) = \xi(\theta)$, $\theta \in [-\tau, 0]$, where $\xi \in \mathcal{C}$. We assume that

(A3) $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the assumption (A1);

(A4) $h : \mathbb{R}^n \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and there exists $L_3 > 0$ such that

$$|h(x_1, y_1, u) - h(x_2, y_2, u)| \leq (L_3|x_1 - x_2| + V_3(y_1, y_2)|y_1 - y_2|)|u|$$

for $x_i, y_i \in \mathbb{R}^n$, $i = 1, 2$, and $u \in U$, where $V_3 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$(3.2) \quad V_3(x, y) \leq K_3(1 + |x|^{q_3} + |y|^{q_3})$$

for some $K_3 > 0$, $q_3 \geq 1$ and arbitrary $x, y \in \mathbb{R}^n$.

Remark 3.1. The jump coefficient may be also *highly nonlinear* with respect to the delay arguments, e.g., for $x, y \in \mathbb{R}$, $u \in U$ and $q > 1$, $h(x, y, u) = y^q u$ satisfies (A4).

Fix $T > 0$ and let the stepsize Δ be defined by (2.3). The EM scheme associated with Eq. (3.1) is defined as follows:

$$(3.3) \quad dY(t) = b(\tilde{Y}(t), \tilde{Y}(t - \tau))dt + \int_U h(\tilde{Y}(t), \tilde{Y}(t - \tau), u)\tilde{N}(dt, du),$$

where $\bar{Y}(t) := Y(k\Delta)$ for $t \in [k\Delta, (k+1)\Delta)$, $k = 0, 1, \dots, M-1$, and $\bar{Y}(\theta) = \xi(\theta)$, $\theta \in [-\tau, 0]$.

To reveal the convergence order of EM scheme (3.3), we need two auxiliary lemmas, where the first one is Bichteler-Jacod inequality for Poisson integrals, e.g., [11, Lemma 3.1].

Lemma 3.1. Let $\Phi : \mathbb{R}_+ \times U \rightarrow \mathbb{R}^n$ and assume that

$$\int_0^t \int_U \mathbb{E}|\Phi(s, u)|^p \lambda(du)ds < \infty, \quad t \geq 0, \quad p \geq 2.$$

Then there exists $D(p) > 0$ such that

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq s \leq t} \left| \int_0^s \int_U \Phi(r, u)\tilde{N}(du, ds) \right|^p\right) &\leq D(p) \left\{ \mathbb{E}\left(\int_0^t \int_U |\Phi(s, u)|^2 \lambda(du)ds\right)^{\frac{p}{2}} \right. \\ &\quad \left. + \mathbb{E} \int_0^t \int_U |\Phi(s, u)|^p \lambda(du)ds \right\}. \end{aligned}$$

Using the Lemma above and the similar argument of Lemma 2.1, we have

Lemma 3.2. Let (A3) and (A4) hold. Then Eq.(3.1) has a unique global solution $(X(t))_{t \in [0, T]}$. Moreover, for any $p \geq 2$ there exists $C > 0$ such that

$$(3.4) \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C,$$

and

$$(3.5) \quad \mathbb{E}|Y(t) - \bar{Y}(t)|^p \leq C\Delta.$$

Remark 3.2. We remark that for $p \geq 2$ all p th-moments of $Y(t) - \bar{Y}(t)$ are bounded by Δ up to a constant, which is completely different from the Brownian motion case (2.6). This is due to the fact that all moments of the increment $\tilde{N}((0, (i+1)\Delta], du) - \tilde{N}((0, i\Delta], du)$ have order $O(\Delta)$ for $\Delta \in (0, 1)$.

We now state our main result in this section.

Theorem 3.3. Let (A3) and (A4) hold. For any $p \geq 2$ and arbitrary $\theta, \alpha \in (0, 1)$, there exists $C > 0$, independent of Δ , such that

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |X(s) - Y(s)|^p \right) \leq C \Delta^{\frac{1}{(1+\theta)[T/\tau](1+\alpha)}}.$$

Proof. The proof of Theorem 3.3 is similar to that of Theorem 2.2, while we give a sketch of the proof to highlight the differences between the Brwonian motion case. Set

$$Z(t) := X(t) - Y(t), \quad \bar{Z}(t) := Y(t) - \bar{Y}(t), \quad \tilde{Z}(t) := (X(t), \bar{Y}(t)) \in \mathbb{R}^{2n}, \quad t \in [0, T].$$

Define for $t \in [0, T]$

$$\Gamma_1(t) := b(X(t), X(t - \tau)) - b(\bar{Y}(t), \bar{Y}(t - \tau))$$

and

$$\Gamma_2(t, u) := h(X(t), X(t - \tau), u) - h(\bar{Y}(t), \bar{Y}(t - \tau), u).$$

For $V_{\delta\epsilon} \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, defined by (2.13), the Itô formula and the Taylor expansion give that for $t \in [0, T]$

$$\begin{aligned} V_{\delta\epsilon}(Z(t)) &= \int_0^t \langle (V_{\delta\epsilon})_x(Z(s)), \Gamma_1(s) \rangle ds + \int_0^t \int_U \{V_{\delta\epsilon}(Z(s) + \Gamma_2(s, u)) \\ &\quad - V_{\delta\epsilon}(Z(s)) - \langle (V_{\delta\epsilon})_x(Z(s)), \Gamma_2(s, u) \rangle\} \lambda(du) ds \\ &\quad + \int_0^t \int_U \{V_{\delta\epsilon}(Z(s) + \Gamma_2(s, u)) - V_{\delta\epsilon}(Z(s))\} \tilde{N}(du, ds) \\ (3.6) \quad &= \int_0^t \langle (V_{\delta\epsilon})_x(Z(s)), \Gamma_1(s) \rangle ds \\ &\quad + \int_0^t \int_U \left\{ \int_0^1 \langle (V_{\delta\epsilon})_x(\theta \Gamma_2(s, u) + Z(s)) - (V_{\delta\epsilon})_x(Z(s)), \Gamma_2(s, u) \rangle d\theta \right\} \lambda(du) ds \\ &\quad + \int_0^t \int_U \left\{ \int_0^1 \langle (V_{\delta\epsilon})_x(\theta \Gamma_2(s, u) + Z(s)), \Gamma_2(s, u) \rangle d\theta \right\} \tilde{N}(du, ds). \end{aligned}$$

By (3.6), together with (2.11) and (2.14), we then deduce that

$$\begin{aligned} |Z(t)| &\leq \epsilon + V_{\delta\epsilon}(Z(t)) \\ &\leq \epsilon + \int_0^t |\Gamma_1(s)| ds + 2 \int_0^t \int_U |\Gamma_2(s, u)| \lambda(du) ds \\ &\quad + \int_0^t \int_U \left\{ \int_0^1 \langle (V_{\delta\epsilon})_x(\theta \Gamma_2(s, u) + Z(s)), \Gamma_2(s, u) \rangle d\theta \right\} \tilde{N}(du, ds), \quad t \in [0, T]. \end{aligned}$$

Furthermore, note from (2.2), (3.2) and (3.4) that for any $q \geq 2$

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} V_1^q(\tilde{Z}(s - \tau))\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} V_3^q(\tilde{Z}(s - \tau))\right) \leq C.$$

Consequently, for any $p \geq 2$ and $t \in [0, T]$, using (2.14) and (3.5), Lemma 3.1 and the Hölder inequality, (A3) and (A4), we derive at

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq s \leq t} |Z(s)|^p\right) &\leq 2^{p-1}(\epsilon^p + \mathbb{E}\left(\sup_{0 \leq s \leq t} V_{\delta\epsilon}^p(Z(s))\right)) \\ &\leq C\left\{\epsilon^p + \int_0^t \mathbb{E}|\Gamma_1(s)|^p ds + \int_0^t \int_U \mathbb{E}|\Gamma_2(s, u)|^p \lambda(du) ds \right. \\ &\quad \left. + \mathbb{E}\left(\int_0^t \int_U |\Gamma_2(s, u)|^2 \lambda(du) ds\right)^{\frac{p}{2}}\right\} \\ &\leq C\left\{\epsilon^p + \int_0^t \mathbb{E}|\Gamma_1(s)|^p ds + \int_0^t \int_U \mathbb{E}|\Gamma_2(s, u)|^p \lambda(du) ds\right\} \\ &\leq C\left\{\epsilon^p + \int_0^t \mathbb{E}(|X(s) - \bar{Y}(s)| \right. \\ &\quad \left. + V_1(\tilde{Z}(s - \tau))|X(s - \tau) - \tilde{Y}(s - \tau)|)^p ds + \int_0^t \mathbb{E}(|X(s) - \bar{Y}(s)| \right. \\ &\quad \left. + V_3(\tilde{Z}(s - \tau))|X(s - \tau) - \tilde{Y}(s - \tau)|)^p ds\right\} \\ &\leq C\left\{\epsilon^p + \Delta + \int_0^t \left\{\mathbb{E}|Z(s)|^p + \mathbb{E}(V_1^p(\tilde{Z}(s - \tau))|Z(s - \tau)|^p) \right. \right. \\ &\quad \left. \left. + \mathbb{E}(V_1^p(\tilde{Z}(s - \tau))|\bar{Z}(s - \tau)|^p) + \mathbb{E}(V_3^p(\tilde{Z}(s - \tau))|Z(s - \tau)|^p) \right. \right. \\ &\quad \left. \left. + \mathbb{E}(V_3^p(\tilde{Z}(s - \tau))|\bar{Z}(s - \tau)|^p)\right\} ds\right\} \\ &\leq C\left\{\epsilon^p + \Delta + \int_0^t \mathbb{E}|Z(s)|^p ds \right. \\ &\quad \left. + \int_0^t \left\{\left(\mathbb{E}|Z(s - \tau)|^{p(1+\theta)}\right)^{\frac{1}{1+\theta}} + \left(\mathbb{E}|\bar{Z}(s - \tau)|^{p(1+\theta)}\right)^{\frac{1}{1+\theta}}\right\} ds\right\}, \end{aligned}$$

where $\theta \in (0, 1)$ is an arbitrary constant. An application of the Gronwall inequality then gives that

$$(3.7) \quad \mathbb{E}\left(\sup_{0 \leq s \leq t} |Z(s)|^p\right) \leq C\left\{\Delta + \int_0^t \left\{\left(\mathbb{E}|Z(s - \tau)|^{p(1+\theta)}\right)^{\frac{1}{1+\theta}} \right. \right. \\ \left. \left. + \left(\mathbb{E}|\bar{Z}(s - \tau)|^{p(1+\theta)}\right)^{\frac{1}{1+\theta}}\right\} ds\right\}, \quad t \in [0, T]$$

by taking $\epsilon = \Delta^{\frac{1}{p}}$. For $\theta \in (0, 1)$ in (3.7) and any $\alpha \in (0, 1)$, let

$$p_i := p(1 + \theta)^{([T/\tau] + 1 - i)(1 + \alpha)}, \quad i = 1, 2, \dots, [T/\tau] + 1.$$

It is trivial to see that

$$(3.8) \quad (1 + \theta)p_{i+1} < p_i \quad \text{and} \quad p_{[T/\tau]+1} = p, \quad i = 1, 2, \dots, [T/\tau].$$

Noting that $Z(t) = \bar{Z}(t) = 0$ for $t \in [-\tau, 0]$, by (3.7) we clearly get

$$\mathbb{E} \left(\sup_{0 \leq s \leq \tau} |Z(s)|^{p_1} \right) \leq C\Delta.$$

This, together with (3.5), (3.7) and the Hölder inequality, yields that

$$(3.9) \quad \begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq 2\tau} |Z(s)|^{p_2} \right) &\leq C \left\{ \Delta + \Delta^{\frac{1}{1+\theta}} + \int_0^{2\tau} \left(\mathbb{E} |Z(s-\tau)|^{p_2(1+\theta)} \right)^{\frac{1}{1+\theta}} ds \right\} \\ &\leq C \left\{ \Delta + \Delta^{\frac{1}{1+\theta}} + \int_0^{2\tau} \left(\mathbb{E} |Z(s-\tau)|^{p_1} \right)^{\frac{p_2}{p_1}} ds \right\} \\ &\leq C \left\{ \Delta + \Delta^{\frac{1}{1+\theta}} + \Delta^{\frac{p_2}{p_1}} \right\} \\ &\leq C\Delta^{\frac{p_2}{p_1}}, \end{aligned}$$

where the last step is due to (3.8). Similarly, we have from (3.7)-(3.9) that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq 3\tau} |Z(s)|^{p_3} \right) &\leq C \left\{ \Delta + \Delta^{\frac{1}{1+\theta}} + \int_0^{3\tau} \left(\mathbb{E} |Z(s-\tau)|^{p_3(1+\theta)} \right)^{\frac{1}{1+\theta}} ds \right\} \\ &\leq C \left\{ \Delta + \Delta^{\frac{1}{1+\theta}} + \int_0^{3\tau} \left(\mathbb{E} |Z(s-\tau)|^{p_2} \right)^{\frac{p_3}{p_2}} ds \right\} \\ &\leq C \left\{ \Delta + \Delta^{\frac{1}{1+\theta}} + \Delta^{\frac{p_3}{p_1}} \right\} \\ &\leq C\Delta^{\frac{p_3}{p_1}}. \end{aligned}$$

Following the previous procedures gives that

$$\mathbb{E} \left(\sup_{0 \leq s \leq T} |Z(s)|^p \right) \leq C\Delta^{\frac{1}{(1+\theta)[T/\tau](1+\alpha)}},$$

and the proof is therefore complete. \square

Remark 3.3. By Theorem 3.3, with $p \geq 2$ increasing the convergence rate of EM scheme (3.3) is decreasing, which is quite different from the Brownian motion case with a constant order $\frac{1}{2}$, and it is therefore best to use the mean-square convergence for the jump case. On the other hand, we reveal that the order of mean-square convergence is close to $\frac{1}{2}$ although the jump diffusion may be highly nonlinear with respect to the delay variables.

References

- [1] Bao, J., Bötcher, B., Mao, X., Yuan, C., Convergence rate of numerical solutions to SFDEs with jumps, *J. Comput. Appl. Math.*, **236** (2011), 119–131.

- [2] Gyöngy, I. and Rásonyi, M., A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients, *Stochastic Process. Appl.*, **121** (2011), 2189–2200.
- [3] Gyöngy, I., A note on Euler’s approximations, *Potential Anal.*, **8** (1998), 205–216.
- [4] Higham, D. J., Kloeden, P. E., Numerical methods for nonlinear stochastic differential equations with jumps, *Numer. Math.*, **101** (2005), 101–119.
- [5] Higham, D. J., Mao, X., Stuart, A. M., Strong convergence of Euler-type methods for nonlinear stochastic differential equations, *SIAM J. Numer. Anal.*, **40** (2002), 1041–1063.
- [6] Hu, Y., Semi-implicit Euler-Maruyama scheme for stiff stochastic equations, *The Silvri Workshop, Progr. Probab. 38, H. Koerezlioglu, ed.*, Birkhauser, Boston **43** (1996), 183–202.
- [7] Jacob, N., Wang, Y., Yuan, C., Numerical solutions of stochastic differential delay equations with jumps, *Stoch. Anal. Appl.*, **27** (2009), 825–853.
- [8] Kloeden, P.E., Platen, E., *Numerical Solution of Stochastic Differential equations*, Springer, Berlin, 3rd edn., 1999.
- [9] Küchler, U., Platen, E., Strong discrete time approximation of stochastic differential equations with time delay, *Math. Comput. Simulation*, **54** (2000), 189–205.
- [10] Mao, X., Sabanis, S., Numerical solutions of stochastic differential delay equations under local Lipschitz condition, *J. Comput. Appl. Math.*, **151** (2003), 215–227.
- [11] Marinelli, C., Prévôt, C., Röckner, M., Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise, *J. Funct. Anal.*, **258** (2010), 616–649.
- [12] Platen, E., Bruti-Liberati, N., *Numerical solution of stochastic differential equations with jumps in finance*, Springer-Verlag, Berlin, 2010.
- [13] Schurz, H., *Stability, stationarity, and boundedness of some implicit numerical methods for stochastic differential equations and applications*, Logos Verlag Berlin, Berlin, 1997.
- [14] Wu, F., Mao, X. and Chen, K., The Cox-Ingersoll-Ross model with delay and strong convergence of its Euler-Maruyama approximate solutions, *Appl. Numer. Math.*, **59** (2009), 2641–2658.
- [15] Yuan, C., Mao, X., A Note on the Rate of Convergence of the Euler-Maruyama Method for Stochastic Differential Equations, *Stoch. Anal. Appl.*, **26** (2008), 325–333.